Detecting Minimal Functions on the Poincaré homology sphere via the heat equation

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Outline

Conjecture

M will be a smooth closed (compact and without boundary) connected manifold; if g is a Riemannian metric for M, Δ_g will denote the Laplace-Beltrami operator (Chavel, 1984),

$$\lambda_0 = 0 < \lambda_1 < \lambda_2 < \dots$$

its eigenvalues, and E_{λ_k} the (real) vector space formed by all eigenfunctions corresponding to λ_k.

Conjecture

Tentative form of the conjecture:
Let (M, g) be a Riemannian manifold such that g is locally homogeneous, i.e. for each pair of points $p, q \in M$ there exist neighborhoods U_p and U_q of p and q respectively, such that there exists an isometry from (U_p, g) to (U_q, g) sending p to q. Then for almost all $f \in L^2(M, g)$, if $u(x, t)$ is the solution to the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} & = \triangle_g u \\ u\left(\cdot, 0\right) & = f \end{cases}$$

there exists $T_f > 0$ such that if $t \geq T_f$ then $u_t := u(., t) : M \to \mathbb{R}$ is Morse and minimal.

Morse function: smooth real valued function whose critical points are nondegenerate.

Minimal Morse function: Morse function having less or the same number of critical points than any other Morse function.

Conjecture

Theorem. (Sufficient condition for the conjecture to hold)
Let (M, g) be a Riemannian manifold where g is arbitrary. If almost every $h \in E_{\lambda_1}$ is Morse and minimal, then the conjecture holds for (M, g).

Note: Almost every $h \in E_{\lambda_1}$: means that h belongs to the complement of a subset of E_{λ_1} which is the union of a countable collection of nowhere dense sets. A set is nowhere dense if the interior of its closure is empty.

Conjecture

We want to *gain evidence* that this sufficient condition holds when (M, g) is $(S^3/I^*, \hat{g}_{S^3})$, i.e. the Poincaré's homology sphere with the spherical metric.

It is known that a Morse function on S^3/I^* is minimal if and only if it has 6 critical points.

To do this we need an explicit basis for E_{λ_1}, and to see that in a random choice of linear combinations all functions are Morse and have 6 critical points.

Spherical Poincaré homology sphere

SPHS

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Binary Icosahedral Group

Let $\mathbb{R}^4 = \{(x, y, z, w)/x, y, z, w \in \mathbb{R}\}$ and let
$S^3 = \left\{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\right\}$
The set

$$\mathbb{H} = \{x + yi + zj + wk : x, y, z, w \in \mathbb{R}\}$$

endowed with the obvious addition and multiplication determined by $i^2 = j^2 = k^2 = -1$, $ij = k, jk = i, ki = j$, $ji = -k, kj = -i, ik = -j$, is the (noncommutative) right of quaternions. Let $q = x + yi + zj + wk$. Its conjugate $\overline{q}$ is $x - yi - zj - wk$ and its norm

$$|q| := \sqrt{q\overline{q}} = \sqrt{x^2 + y^2 + z^2 + w^2}$$

Binary Icosahedral Group

A quaternion q is called unitary if $|q| = 1$. The set of unitary quaternions with quaternion multiplication is a noncommutative group.
Clearly $\mathbb{R}^4$ can be identified with $\mathbb{H}$ via

$$(x, y, z, w) \to x + yi + zj + wk$$

This identification preserves addition, multiplication by a scalar, and norm, and it identifies S^3 with the set of unitary quaternions

$$\{q \in \mathbb{H} / |q| = 1\}$$

Binary Icosahedral Group

Now, $S^3 \subset \mathbb{R}^4$ is a smooth, closed, connected, 3 dimensional manifold, which inherits a Riemannian metric g_{S^3} from the euclidean metric of its ambient space $\mathbb{R}^4$.

The identification of S^3 with the unitary quaternions allows us to define, for every $q \in S^3$, a map $T_q : S^3 \to S^3$ as $T_q(p) = qp$.
It can be seen that each T_q is an *isometry* of (S^3, g_{S^3}).

Binary Icosahedral Group

There is a subgroup of the group of unitary quaternions, the binary icosahedral group I^* that we describe next. I^* is formed by 120 elements:

- The 16 quaternions $\pm\frac{1}{2} \pm \frac{1}{2}i \pm \frac{1}{2}j \pm \frac{1}{2}k$.
- The 8 quaternions obtained from $0 + 0i + 0j \pm k$ by taking all permutations of its coefficients.
- And the 96 quaternions obtained by taking all even permutations of the coefficients of the quaternion $\pm 1 \pm \varphi i \pm \varphi^{-1} j + 0k$.

where $\varphi = \left(1 + \sqrt{5}\right)/2$ is the golden proportion.

Binary Icosahedral Group

We can define a left action of I^* on S^3

$$a : I^* \times S^3 \to S^3, a\left(q, p\right) = qp$$

The left action a satisfies the conditions for the set of orbits S^3/I^* to be a smooth, closed, connected, 3-dimensional manifold endowed with a Riemannian metric $\hat{g}_{S^3}$.
$(S^3/I^*, \hat{g}_{S^3})$ is called the Spherical Poincaré homology sphere. It is homologically indistinguishable from S^3!

Fundamental Domain

A very effective way to get a concrete picture of a quotient space, like S^3/I^*, is to determine a **fundamental domain** D and the identification of points on its boundary ∂D induced by the action.

Let (X, g_X) be a Riemannian manifold and let d_X be the metric on X induced by g_X. Let $a : G \times X \to X$ be a left action of a group G on X. For each $g \in G$ we define the map $T_g : X \to X$ sending x to $a(g, x)$. We require that a "acts by isometries", i.e. that for each $g \in G$, T_g an isometry.

If a has certain properties, the set of orbits $\{\{a(g, x) : g \in G\} : x \in X\}$, denoted X/G is a Riemannian manifold.

Fundamental Domain

A **fundamental domain** D is the closure $\overline{E}$ of a $E \subset X$, usually asked to be connected, containing exactly one element of each orbit of the action. The sets $g.E := \{a(g,x)/x \in E\}$ are such that $g_1.E \cap g_2.E = \emptyset$ whenever $g_1 \neq g_2$, and $X = \underset{g \in G}{\cup} g.E$.

Although the analogously defined sets $g.D$, $g \in G$ do not form a partition of X, in important contexts E can be adequately chosen so that if $g_1 \neq g_2$, $g_1.D$ and $g_2.D$ only meet along their boundaries, i.e. $g_1.D \cap g_2.D$ is a subset of $\partial(g_1.D) \cap \partial(g_2.D)$. When this happens one says that $\{g.D/g \in G\}$ is a *tessellation* of X.

It can be seen that if we define a binary relation on ∂D by $x \sim y$ if there is a $g \in G$ with $a(g,x) = y$ then the relation is an equivalence relation and $D/\sim$ is isometric to X/G. This allows one to think of X/G as obtained by taking D and glueing parts of its boundary to other parts, according to the action.

Fundamental Domain

There is a particularly nice way to pick a fundamental domain.

If $p \in X$, the set

$$D(p) = \{q \in X / \forall g \in G, d_X(p,q) \leq d_X(p, a(g,q))\}$$

is a fundamental domain called the Dirichlet fundamental domain. It can be seen that $\{g.D(p)/g \in G\}$ is a tessellation of X, where by definition any two tiles $g_1.D(p)$ and $g_2.D(p)$ are isometric. It is easy to see that $D(p)$ equals

$$V(p) = \{q \in X / \forall g \in G, d_X(p,q) \leq d_X(a(g,p), q)\}$$

This set is called the Voronoi cell of p respect to the orbit of p.

Fundamental Domain

For example, if $X = (\mathbb{R}, g_{\mathbb{R}})$ and $G = (\{2\pi n / n \in \mathbb{Z}\}, +)$, then $a(2\pi n, x) = x + 2\pi n$ is an action by isometries, and for each $p \in \mathbb{R}$, then $D(p) = V(p) = [p - \pi, p + \pi]$ is a fundamental domain of this action. Now X/G is isometric to $D(p)/\sim$ and this is the interval $[p - \pi, p + \pi]$ with its ends identified, giving a circle.

For $u, v \in S^3$, $d_{S^3}(u, v) = \arccos(u \cdot v)$ where $u \cdot v$ is the usual inner product in $\mathbb{R}^4$.
Since the action of I^* on S^3 is by isometries, for each $p \in S^3$ the set

$$V(p) = \left\{x \in S^3 : d_{S^3}(x, p) \leq d_{S^3}(x, qp), \text{for all } q \in I^*\right\}$$

is a fundamental domain.

Fundamental Domain

We now describe $V(p)$ where $p = (0,0,0,1)$ is the North of S^3, so that we can visualize it and use it in an algorithm for detecting critical points of functions defined on S^3/I^*.

We first calculate the orbit of p, $O = \{a(g,p)/g \in S^3\}$. This set has 120 elements and p is one of them. In this set there are exactly 12 points q such that $d_{S^3}(q,p)$ is $\min\{d_{S^3}(q,p)/q \in O - \{p\}\}$.

To visualize these constructions in $\mathbb{R}^3$ we use the stereographic projection of S^3 from its south pole $s = (0,0,0,-1)$ defined as

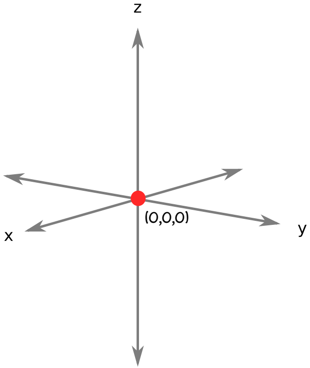
$$\pi_s : S^3 \setminus \{s\} \to \mathbb{R}^3$$

$$\pi_s\,(x,y,z,w) = \left(\frac{x}{1+w}, \frac{y}{1+w}, \frac{z}{1+w}\right)$$

Fundamental Domain

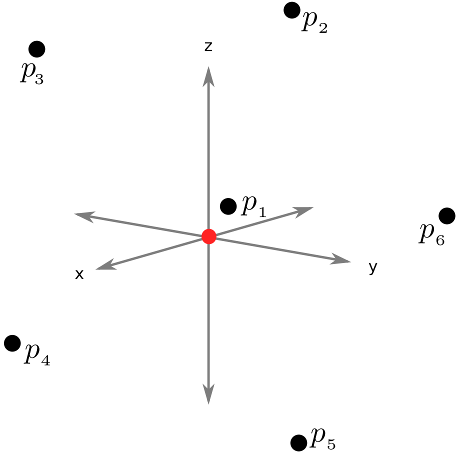
Notice that the stereographic projection of the center $p = (0,0,0,1)$ of the fundamental domain is

$$\pi_s\,(0,0,0,1) = \left(\frac{0}{1+1}, \frac{0}{1+1}, \frac{0}{1+1}\right) = (0,0,0)$$



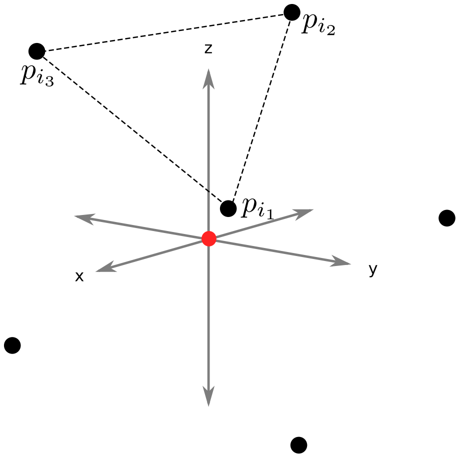
Fundamental Domain

The projection of 6 of the 12 points $p_1, \ldots, p_{12}$



Fundamental Domain

Let $\delta = \min\{d_{S^3}(p_i, p_j) : i \neq j\}$. It turns out that there are exactly 20 triads $\{p_{i_1}, p_{i_2}, p_{i_3}\}$ such that $\delta = d(p_{i_1}, p_{i_2}) = d(p_{i_1}, p_{i_3}) = d(p_{i_2}, p_{i_3})$.



Fundamental Domain

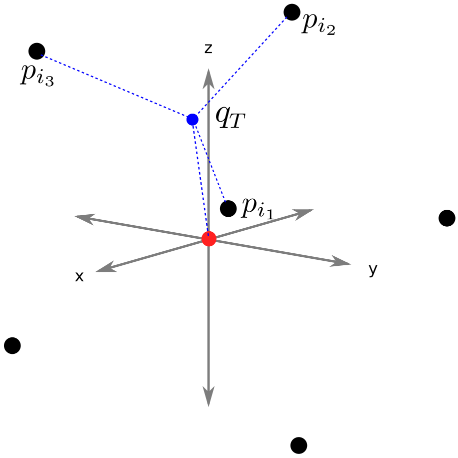
For each triad $T = \{p_{i_1}, p_{i_2}, p_{i_3}\}$ we consider the three hyperplanes H_1^T, H_2^T, H_3^T in $\mathbb{R}^4$ through the origin $(0,0,0,0)$ and whose respective normal vectors n_1^T, n_2^T, n_3^T are the differences $p - p_{i_1}, p - p_{i_2}, p - p_{i_3}$.

The intersection

$$H_1^T \cap H_2^T \cap H_3^T \cap S^3$$

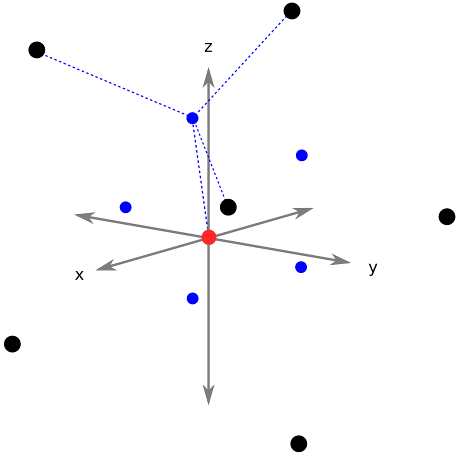
consists of one point q_T. This point q_T is equidistant to $p_{i_1}, p_{i_2}, p_{i_3}$ and p.

The projection of this point is depicted in



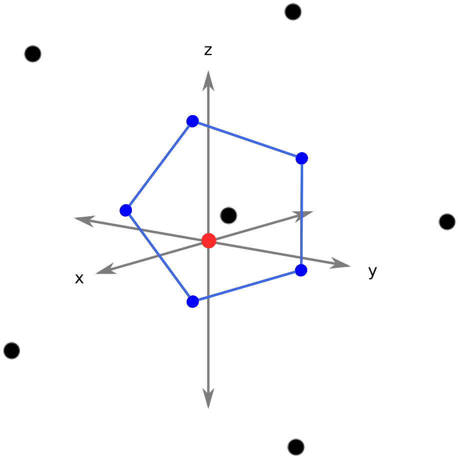
Fundamental Domain

Each point p_i lies in exactly five triads $T_1^i, \ldots, T_5^i$.



Fundamental Domain

The projection of the points $q_{T_1^i}, \ldots, q_{T_5^i}$ are the vertices of a pentagonal face, which is the projection of a face of $V(p)$.



Fundamental Domain

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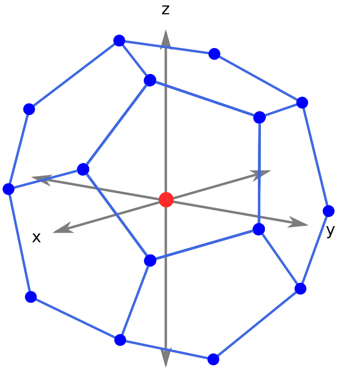

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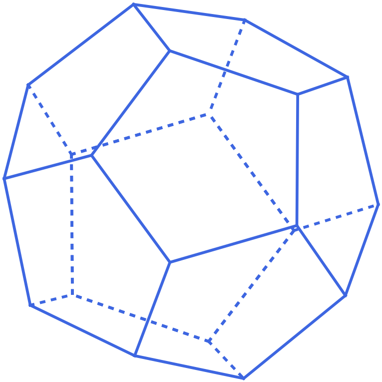
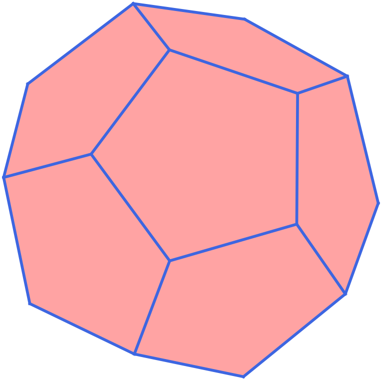


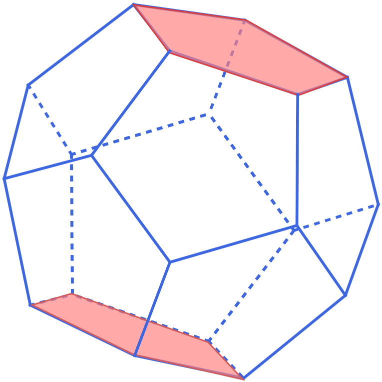
Fundamental Domain

The projection of the fundamental domain is therefore a solid dodecahedron whose faces are slightly inflated.



Fundamental Domain

It can be seen that the Poincaré homology sphere is homeomorphic to the space obtained by glueing the opposite faces of this solid dodecahedron, according to the rule depicted in



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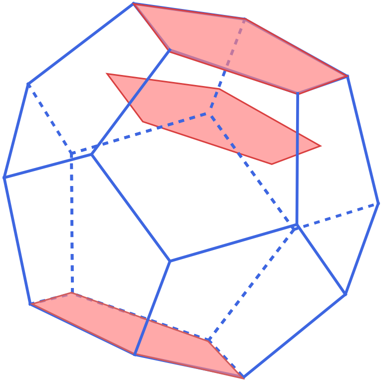

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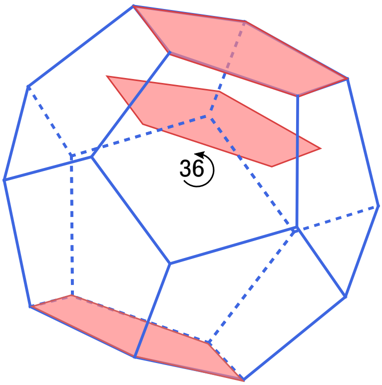

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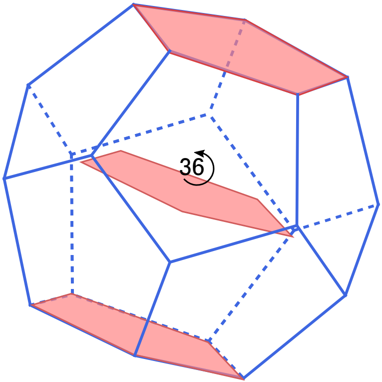

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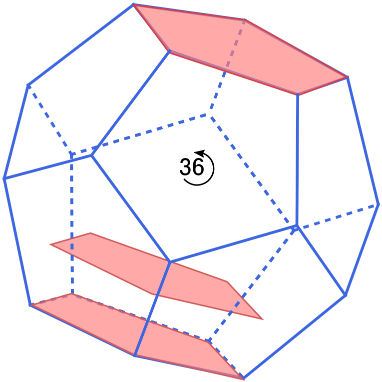

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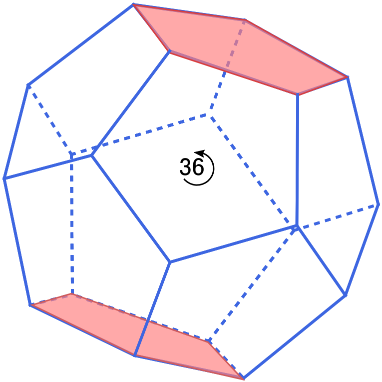

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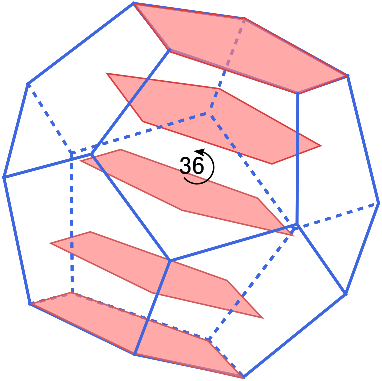

Eigenvalues and Eigenfunctions of the Laplace-Beltrami operator on the Spherical Poincaré homology sphere

Eigenfunctions

The paper (Weeks, 2006) gives an explicit description of the eigenvalues and *complex valued eigenfunctions* of the Laplace-Beltrami operator on the Spherical Poincaré homology sphere.

First, identify $\mathbb{R}^4$ with $\mathbb{C}^2 = \{(\alpha, \beta)/\alpha, \beta \in \mathbb{C}\}$, by identifying (x, y, z, w) with $(x + yi, z + wi)$. We are only interested in the space E_{λ_1} of the SPHS.

According to (Weeks, 2006), $\lambda_1 = 12$ and the complex vector space $E_{\lambda_1}^{\mathbb{C}} := \{f : \mathrm{SPHS} \to \mathbb{C}/\Delta f = \lambda_1 f\}$ has complex dimension 13 and a basis can be obtained as follows.

Eigenfunctions

Let

$$F_1 = \alpha^{11}\beta + 11\alpha^6\beta^6 - \alpha\beta^{11}$$

This function is a complex valued function defined on $\mathbb{C}^2$. Now we define the operator

$$twist_-(F) = -\overline{\beta}\frac{\partial F}{\partial \alpha} + \overline{\alpha}\frac{\partial F}{\partial \beta}$$

Eigenfunctions

Applying this operator successively 12 times we obtain the thirteen functions

$$
\begin{aligned}
F_1 &= \alpha^{11}\beta + 11\alpha^6\beta^6 - \alpha\beta^{11} \\
F_2 &= twist_-(F_1) \\
F_3 &= twist_-(F_2) \\
&\vdots \\
F_{13} &= twist_-(F_{12})
\end{aligned}
$$

The *descent to SPHS of the restrictions to* $S^3 \subset \mathbb{C}^2$ of the functions $F_1, \dots, F_{13}$ constitute a basis for the space

$$E^{\mathbb{C}}_{\lambda_1} := \{f : SPHS \to \mathbb{C} / \Delta f = \lambda_1 f\}$$

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Eigenfunctions

It can be seen that the *descent to SPHS of the restrictions to* $S^3 \subset \mathbb{C}^2$ of the functions $f_1, \ldots, f_{13}$ defined as

$$\{f_1 = Re(F_1), f_2 = Im(F_1), f_3 = Re(F_2), f_4 = Im(F_2), f_5 = Re(F_3),$$
$$f_6 = Im(F_3), f_7 = Re(F_4), f_8 = Im(F_4), f_9 = Re(F_5),$$
$$f_{10} = Im(F_{6)}, f_{11} = Re(F_6), f_{12} = Im(F_6), f_{13} = Re(F_7)\}$$

is a basis for the space E_{λ_1} of real valued eigenfunctions of $\lambda_1 = 12$ on SPHS.

Eigenfunctions

For example:

$$
\begin{aligned}
f_1 = & -w^{11}y + 11w^{10}xz + 55w^9yz^2 - 165w^8xz^3 - 330w^7yz^4 - 11 \\
& w^6x^6 + 165w^6x^4y^2 - 165w^6x^2y^4 + 462w^6xz^5 + 11w^6 \\
& y^6 - 396w^5x^5yz + 1320w^5x^3y^3z - 396w^5xy^5z + 462w^5y \\
& z^6 + 165w^4x^6z^2 - 2475w^4x^4y^2z^2 + 2475w^4x^2y^4z^2 - \\
& 330w^4xz^7 - 165w^4y^6z^2 + 1320w^3x^5yz^3 - 4400w^3x^3y^3 \\
& z^3 + 1320w^3xy^5z^3 - 165w^3yz^8 - 165w^2x^6z^4 + 2475w^2 \\
& x^4y^2z^4 - 2475w^2x^2y^4z^4 + 55w^2xz^9 + 165w^2y^6z^4 - \\
& 11wx^{10}y + 165wx^8y^3 - 462wx^6y^5 - 396wx^5yz^5 + 330wx^4 \\
& y^7 + 1320wx^3y^3z^5 - 55wx^2y^9 - 396wxy^5z^5 + wy^{11} + 11wy \\
& z^{10} + x^{11}z - 55x^9y^2z + 330x^7y^4z + 11x^6z^6 - 462x^5y^6z - 165x^4y^2z^6 \\
& + 165x^3y^8z + 165x^2y^4z^6 - 11xy^{10}z - xz^{11} - 11y^6z^6
\end{aligned}
$$

Application of Algorithms and Results

We now want to take random real linear combinations

$$f = a_1 f_1 + \ldots + a_{13} f_{13}$$

and either

i) determine the number of critical points of their restrictions to S^3 (dividing the resulting number by 120 gives the number of critical points of the descent of the corresponding function to SPHS);

or, equivalently

ii) determine the number of critical points of the function $f \circ \pi_s^{-1} : \mathbb{R}^3 \to \mathbb{R}$ belonging to $\pi_s(V(p))$.

Application of Algorithms and Results

Frank Sottile followed approach *i)*. He took the function f_1 and using Software Bertini, found that $f_1|_{S^3}$ has 1440 critical points. So this function has $1440/120 = 12$ critical points in SPHS. We expected 6.

I followed approach *ii)*. I run the algorithm for Detecting Critical Regions (Weber, Scheuermann, Hagen, and Hamann 2003) presented in my last talk, followed by the nsolve function of the Python SymPy library.
With the following results:

f_1, f_2, f_{12} and f_{13} have 12 critical points
$f_3, f_4, \ldots, f_{11}$ have 6 critical points
$f_1 + f_3$ has 6 critical points
$f_4 + f_7$ has 6 critical points
$f_2 + f_3 + f_4$ has 6 critical points
$5f_2 + 2f_3 - 7f_5$ has 6 critical points
$-3f_6 - 2f_8 + 4f_{11}$ has 6 critical points

Thanks!

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